

# Expectation values of descendent fields in the Bullough-Dodd model and related perturbed conformal field theories

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## Abstract

The exact vacuum expectation values of the second level descendent fields  $\langle (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{a\varphi} \rangle$  in the Bullough-Dodd model are calculated. By performing quantum group restrictions, we obtain  $\langle L_{-2}\bar{L}_{-2}\Phi_{lk} \rangle$  in the  $\Phi_{12}$ ,  $\Phi_{21}$  and  $\Phi_{15}$  perturbed minimal CFTs. In particular, the exact expectation value  $\langle T\bar{T} \rangle$  is found to be proportional to the square of the bulk free energy.

## 1 Introduction

In a 2-D integrable quantum field theory (QFT) which can be realized as a conformal field theory (CFT) perturbed by some relevant operator, it is well-known that any correlation function of local fields  $\mathcal{O}_a(x)$  in the short-distance limit can be reduced down to one-point functions  $\langle \mathcal{O}_{a'}(x) \rangle$  by successive application of the operator product expansion (OPE) [1, 2]. These vacuum expectation values (VEV)s contain important information about the IR environment. Together with the structure constants characterizing the UV limit of the QFT, they provide the UV behaviour of the correlation functions whereas the so-called form-factors characterize their IR behaviour. Since three years important progress has been made concerning the evaluation of some VEVs in different integrable QFTs. In ref. [3], an explicit expression for the VEVs of the exponential fields in the sinh-Gordon and sine-Gordon models was proposed. In ref. [4, 5] it was shown that this result can be obtained using the “reflection amplitude” [6] of the Liouville field theory. This method was also applied in the so-called Bullough-Dodd model with real and imaginary coupling. In QFT involving more fields, the VEVs for a two-parameter family of integrable QFTs introduced and studied in [7] gave rise to the VEVs of local operators in parafermionic sine-Gordon models and in integrable perturbed  $SU(2)$  coset CFT [8]. Also, the VEVs

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in simply-laced affine Toda field theories are known for a long time [9] and the case of non-simply laced dual pairs was recently studied in [10, 11].

However, the higher-order corrections to the short-distance expansion of two-point correlation functions involve the VEVs of the descendent fields. This question was addressed in [12]. There, the VEV of the descendent field  $\langle (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{a\varphi} \rangle$  in the sinh-Gordon (ShG) and sine-Gordon (SG) - with  $a \rightarrow i\alpha$  - models was calculated as well as in its related perturbed CFT, i.e.  $\Phi_{13}$  perturbation of minimal models. From this result, the next-order correction of the two-point function in the scaling Lee-Yang model was computed [2, 5].

The purpose of this paper is to calculate the VEV of the simplest non-trivial descendent field in the Bullough-Dodd (BD) model which is generally described by the following action in the Euclidean space :

$$\mathcal{A}_{BD} = \int d^2x \left[ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu e^{b\varphi} + \mu' e^{-\frac{b}{2}\varphi} \right]. \quad (1.1)$$

Here, the parameters  $\mu$  and  $\mu'$  are introduced, as the two operators do not renormalize in the same way, on the contrary to any simply-laced affine Toda field theory. This model has attracted over the years a certain interest, in particular in connection with perturbed minimal models :  $c < 1$  minimal CFT perturbed by the operators  $\Phi_{12}$ ,  $\Phi_{21}$  or  $\Phi_{15}$  can be obtained by a quantum group (QG) restriction of imaginary Bullough-Dodd model [14, 15, 16, 17, 5] with special values of the coupling. We will use this property to deduce the VEV  $\langle L_{-2} \bar{L}_{-2} \Phi_{lk} \rangle$  in the following perturbed minimal models :

$$\mathcal{A} = \mathcal{M}_{p/p'} + \lambda \int d^2x \Phi_{12}, \quad (1.2)$$

$$\hat{\mathcal{A}} = \mathcal{M}_{p/p'} + \hat{\lambda} \int d^2x \Phi_{21} \quad (1.3)$$

$$\text{or} \quad \tilde{\mathcal{A}} = \mathcal{M}_{p/p'} + \tilde{\lambda} \int d^2x \Phi_{15}, \quad (1.4)$$

where we denote respectively  $\Phi_{12}$ ,  $\Phi_{21}$  and  $\Phi_{15}$  as specific primary operators of the unperturbed minimal model  $\mathcal{M}_{p/p'}$  and the parameters  $\lambda$ ,  $\hat{\lambda}$  and  $\tilde{\lambda}$  characterize the strength of the perturbation.

This paper is organized as follows. In the next section we introduce the notations and write the short-distance expansion of the two-point correlation function which involves the VEV of the descendent field  $\langle (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{a\varphi} \rangle$  in the BD model associated with the action (1.1). Using the method based on the “reflection relations” [6] we find a conjecture for this last quantity in Section 3. Whereas it exists an ambiguity for the solution of these functional equations, we choose the “minimal” one which is compatible with the “resonance conditions” (see ref. [12] for details). In Section 4 we compare the semi-classical limit of the short distance expansion of the two-point function with the semi-classical approximation based on the action (1.1). In Section 5 we deduce the VEV  $\langle L_{-2} \bar{L}_{-2} \Phi_{lk} \rangle$  in the models (1.2), (1.3) and (1.4). Concluding remarks follow in the last section.

## 2 Short-distance expansion of the two-point function in the BD model

Similarly to the ShG model [12], the BD model can be regarded as a relevant perturbation of a Gaussian CFT. In this free field theory, the field is normalized such that:

$$\langle \varphi(z, \bar{z}) \varphi(0, 0) \rangle_{Gauss} = -2 \log(z\bar{z}). \quad (2.1)$$

and we have the classical equation of motion :

$$\partial \bar{\partial} \varphi = 0. \quad (2.2)$$

Instead of considering the action (1.1) we turn directly to the case of an imaginary coupling constant which is the most interesting for our purpose in Section 5. The perturbation is then relevant if  $0 < \beta^2 < 1$ . Although the model (1.1) for real coupling is very different from the one with imaginary coupling in its physical content (this latter model contains solitons and breathers), there are good reasons to believe that the expectation values obtained in the real coupling case provide also the expectation values for the imaginary coupling. The calculation of the VEVs in both cases ( $b$  real or imaginary) within the standard perturbation theory agree through the identification  $b = i\beta$  [5]. With this substitution in (1.1), the general short distance OPE for two arbitrary primary fields  $e^{i\alpha_1\varphi}(x)$  and  $e^{i\alpha_2\varphi}(y)$  takes the form :

$$\begin{aligned} e^{i\alpha_1\varphi}(x) e^{i\alpha_2\varphi}(y) &= \sum_{n=0}^{\infty} \{ C_{\alpha_1\alpha_2}^{n,0}(r) e^{i(\alpha+n\beta)\varphi}(y) + \dots \} \\ &+ \sum_{n=1}^{\infty} \{ C_{\alpha_1\alpha_2}'^{n,0}(r) e^{i(\alpha-\frac{n\beta}{2})\varphi}(y) + \dots \} \\ &+ \sum_{n=1}^{\infty} \{ D_{\alpha_1\alpha_2}^{n,0}(r) e^{i(\alpha+(n-\frac{1}{2})\beta)\varphi}(y) + \dots \} \end{aligned} \quad (2.3)$$

where  $\alpha = \alpha_1 + \alpha_2$ ,  $r = |x - y|$  and the dots in each term stand for the contributions of the descendents of each field. The different coefficients in eq. (2.3) are computable within the conformal perturbation theory (CPT) [2, 18]. We obtain :

$$\begin{aligned} C_{\alpha_1\alpha_2}^{n,0}(r) &= \mu^n r^{4\alpha_1\alpha_2 + 4n\beta(\alpha_1+\alpha_2) + 2n(1-\beta^2) + 2n^2\beta^2} f_{\alpha_1\alpha_2}^{n,0}(\mu(\mu')^2 r^{6-3\beta^2}); \\ C_{\alpha_1\alpha_2}'^{n,0}(r) &= \mu'^n r^{4\alpha_1\alpha_2 - 2n\beta(\alpha_1+\alpha_2) + 2n(1-\frac{\beta^2}{4}) + \frac{n^2\beta^2}{2}} f_{\alpha_1\alpha_2}'^{n,0}(\mu(\mu')^2 r^{6-3\beta^2}); \\ D_{\alpha_1\alpha_2}^{n,0}(r) &= \mu' \mu^n r^{4\alpha_1\alpha_2 + 4(n-\frac{1}{2})\beta(\alpha_1+\alpha_2) + 2n(1-2\beta^2) + 2 + 2n^2\beta^2} g_{\alpha_1\alpha_2}^{n,0}(\mu(\mu')^2 r^{6-3\beta^2}) \end{aligned} \quad (2.4)$$

where any function  $h \in \{f, f', g\}$  admits a power series expansion :

$$h_{\alpha_1\alpha_2}^{n,0}(t) = \sum_{k=0}^{\infty} h_k^{n,0}(\alpha_1, \alpha_2) t^k. \quad (2.5)$$

Each coefficient in (2.4) is expressed in terms of Coulomb type integrals. The corresponding leading terms are respectively given by :

$$\begin{aligned} f_0^{n,0}(\alpha_1, \alpha_2) &= j_n(\alpha_1\beta, \alpha_2\beta, \beta^2) \quad \text{for } n \neq 0 ; \\ f_0^{m,0}(\alpha_1, \alpha_2) &= j_n\left(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}\right) ; \\ g_0^{n,0}(\alpha_1, \alpha_2) &= \mathcal{F}_{n,1}(\alpha_1\beta, \alpha_2\beta, \beta^2) \end{aligned} \quad (2.6)$$

where we introduced the integrals :

$$\begin{aligned} j_n(a, b, \rho) &= \frac{1}{n!} \int \prod_{k=1}^n d^2x_k \prod_{k=1}^n |x_k|^{4a} |1 - x_k|^{4b} \prod_{k < p}^n |x_k - x_p|^{4\rho} ; \\ \mathcal{F}_{n,m}(a, b, \rho) &= \frac{1}{n!m!} \int \prod_{k=1}^n d^2x_k \int \prod_{l=1}^m d^2y_l \prod_{k=1}^n |x_k|^{4a} |1 - x_k|^{4b} \prod_{k < p}^n |x_k - x_p|^{4\rho} \\ &\quad \times \prod_{l=1}^m |y_l|^{-2a} |1 - y_l|^{-2b} \prod_{l < q}^m |y_l - y_q|^\rho \prod_{k,l}^{n,m} |x_k - y_l|^{-2\rho} . \end{aligned} \quad (2.7)$$

Notice that  $f_0^{0,0}(\alpha_1, \alpha_2) = 1$  and that the first subleading term of the coefficient  $C_{\alpha_1\alpha_2}^{0,0}$  is :

$$f_1^{0,0}(\alpha_1, \alpha_2) = \mathcal{F}_{1,2}(\alpha_1\beta, \alpha_2\beta, \beta^2) . \quad (2.8)$$

The integrals  $j_n(a, b, \rho)$  have been evaluated explicitly in [18] with the result :

$$\begin{aligned} j_n(a, b, \rho) &= \pi^n (\gamma(\rho))^{-n} \prod_{k=1}^n \gamma(k\rho) \times \\ &\quad \prod_{k=0}^{n-1} \gamma(1 + 2a + k\rho) \gamma(1 + 2b + k\rho) \gamma(-1 - 2a - 2b - (n - 1 + k)\rho) \end{aligned} \quad (2.9)$$

where the usual notation  $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$  is used.

As we already said in the introduction, the next sub-leading terms in (2.3) involve the descendent fields. There are four independent second-level descendent fields in BD :

$$\begin{aligned} (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i\alpha\varphi} ; &\quad (\partial\varphi)^2(\bar{\partial}^2\varphi) e^{i\alpha\varphi} ; \\ (\partial^2\varphi)(\bar{\partial}\varphi)^2 e^{i\alpha\varphi} ; &\quad (\partial^2\varphi)(\bar{\partial}^2\varphi) e^{i\alpha\varphi} . \end{aligned} \quad (2.10)$$

Similarly to the SG (or ShG) case, using (2.2) it is easy to show that linear combinations of these descendent fields can be written in terms of total derivatives of local fields (we refer the reader to [12] for details about these relations). As a result, the VEVs of the composite fields (2.10) can all be expressed in terms of a single VEV, say :

$$< (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i\alpha\varphi} >_{BD} . \quad (2.11)$$

Let us make an important observation. The second sub-leading terms in the OPE (2.3) appear to be the third order descendents of the primary fields. Analogously to

the previous discussion linear combinations of them can be expressed in terms of total derivatives of some local fields. As before, all the corresponding VEVs can be expressed through  $\langle (\partial\varphi)^3(\bar{\partial}\varphi)^3 e^{i\alpha\varphi} \rangle$ . Unlike the SG case, it is non-vanishing due to the absence of a conserved charge of spin 3 in the BD model. We will not enter in details about this VEV since its computation is not our purpose in this paper.

One can now write the short-distance expansion for the two-point function :

$$\mathcal{G}_{\alpha_1\alpha_2}(r) = \langle e^{i\alpha_1\varphi}(x)e^{i\alpha_2\varphi}(y) \rangle_{BD} \quad \text{with } r = |x - y| \quad (2.12)$$

by taking the expectation value of the r.h.s. of the OPE (2.3) in the BD model with imaginary coupling. Due to the previous discussion, the first non-vanishing contribution of the VEVs of lowest descendent fields in the r.h.s. of the VEV of (2.3) correspond to the following terms :

$$\begin{aligned} C_{\alpha_1\alpha_2}^{n,2}(r) &< (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i(\alpha+n\beta)\varphi} \rangle_{BD} ; \\ C_{\alpha_1\alpha_2}'^{n,2}(r) &< (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i(\alpha-\frac{n\beta}{2})\varphi} \rangle_{BD} ; \\ D_{\alpha_1\alpha_2}^{n,2}(r) &< (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i(\alpha+(n-\frac{1}{2})\beta)\varphi} \rangle_{BD} , \end{aligned} \quad (2.13)$$

respectively. These coefficients also admit expansion similar to eqs. (2.4), (2.5) and (2.6). In particular we have :

$$C_{\alpha_1,\alpha_2}^{0,2}(r) = \frac{(\alpha_1\alpha_2)^2}{4} r^{4\alpha_1\alpha_2+4} \left( 1 + O(\mu(\mu')^2 r^{6-3\beta^2}) \right). \quad (2.14)$$

Finally, the short-distance ( $r \rightarrow 0$ ) expansion of the two-point correlation function in the BD model with imaginary coupling writes :

$$\begin{aligned} \mathcal{G}_{\alpha_1\alpha_2}(r) &= \mathcal{G}_{\alpha_1+\alpha_2} r^{4\alpha_1\alpha_2} \left\{ 1 + \mathcal{F}_{1,2}(\alpha_1\beta, \alpha_2\beta, \beta^2) \mu(\mu')^2 r^{6-3\beta^2} + \frac{(\alpha_1\alpha_2)^2}{4} \mathcal{H}(\alpha_1 + \alpha_2) r^4 \right. \\ &\quad \left. - \frac{\alpha_1^2 \alpha_2^2 (\alpha_1 - \alpha_2)^2}{144} \mathcal{K}(\alpha_1 + \alpha_2) r^6 + O(\mu^2 (\mu')^4 r^{12-6\beta^2}) \right\} \\ &+ \sum_{n=1}^{\infty} \mu^n r^{4\alpha_1\alpha_2+4n\beta(\alpha_1+\alpha_2)+2n(1-\beta^2)+2n^2\beta^2} j_n(\alpha_1\beta, \alpha_2\beta, \beta^2) \\ &\quad \times \mathcal{G}_{\alpha_1+\alpha_2+n\beta} \left\{ 1 + O(\mu(\mu')^2 r^{6-3\beta^2}) \right\} \\ &+ \sum_{n=1}^{\infty} \mu^n r^{4\alpha_1\alpha_2-2n\beta(\alpha_1+\alpha_2)+2n(1-\frac{\beta^2}{4})+\frac{n^2\beta^2}{2}} j_n\left(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}\right) \\ &\quad \times \mathcal{G}_{\alpha_1+\alpha_2-\frac{n\beta}{2}} \left\{ 1 + O(\mu(\mu')^2 r^{6-3\beta^2}) \right\} \\ &+ \sum_{n=1}^{\infty} \mu^n \mu' r^{4\alpha_1\alpha_2+4(n-\frac{1}{2})\beta(\alpha_1+\alpha_2)+2n(1-2\beta^2)+2+2n^2\beta^2} \mathcal{F}_{n,1}(\alpha_1\beta, \alpha_2\beta, \beta^2) \\ &\quad \times \mathcal{G}_{\alpha_1+\alpha_2+(n-\frac{1}{2})\beta} \left\{ 1 + O(\mu(\mu')^2 r^{6-3\beta^2}) \right\} \end{aligned} \quad (2.15)$$

where we defined  $\mathcal{H}(\alpha)$  and  $\mathcal{K}(\alpha)$  by the ratios :

$$\mathcal{H}(\alpha) = \frac{\langle (\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i\alpha\varphi} \rangle_{BD}}{\langle e^{i\alpha\varphi} \rangle_{BD}}, \quad (2.16)$$

$$\mathcal{K}(\alpha) = \frac{< (\partial\varphi)^3 (\bar{\partial}\varphi)^3 e^{i\alpha\varphi} >_{BD}}{< e^{i\alpha\varphi} >_{BD}} \quad (2.17)$$

and  $\mathcal{G}_\alpha = < e^{i\alpha\varphi} >_{BD}$  is the VEV of the exponential field in the BD model. A closed analytic expression for this latter VEV has been proposed in ref. [5] :

$$\begin{aligned} < e^{i\alpha\varphi} >_{BD} = & \left[ \frac{\mu' 2^{-\frac{\beta^2}{2}} \Gamma(1+\beta^2) \Gamma(1-\frac{\beta^2}{4})}{\mu \Gamma(1-\beta^2) \Gamma(1+\frac{\beta^2}{4})} \right]^{\frac{2\alpha}{3\beta}} \left[ \frac{m \Gamma(1-\frac{\beta^2}{6-3\beta^2}) \Gamma(\frac{2}{6-3\beta^2})}{2^{\frac{2}{3}} \sqrt{3} \Gamma(\frac{1}{3})} \right]^{-\alpha\beta+2\alpha^2} \times \\ & \exp \left[ \int_0^{+\infty} \frac{dt}{t} \left( \frac{\sinh((2-\beta^2)t) \Psi(t, \alpha)}{\sinh(3(2-\beta^2)t) \sinh(2t) \sinh(\beta^2 t)} - 2\alpha^2 e^{-2t} \right) \right] \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} \Psi(t, \alpha) = & -\sinh(2\alpha\beta t) (\sinh((4-\beta^2-2\alpha\beta)t) - \sinh((2-2\beta^2+2\alpha\beta)t) + \\ & \sinh((2-\beta^2-2\alpha\beta)t) - \sinh((2-\beta^2+2\alpha\beta)t) - \sinh((2+\beta^2-2\alpha\beta)t)). \end{aligned}$$

Its integral representation is well defined if :

$$-\frac{1}{2\beta} < \Re(\alpha) < \frac{1}{\beta} \quad (2.19)$$

and obtained by analytic continuation outside this domain.

It is then straightforward to obtain the result associated with the action (1.1) i.e. for *real* values of the coupling constant  $b$  which follows from the obvious substitutions :

$$\begin{aligned} \beta &\rightarrow -ib; & \alpha_1 &\rightarrow -ia_1; & \alpha_2 &\rightarrow -ia_2; \\ \mu &\rightarrow -\mu; & \mu' &\rightarrow -\mu'. \end{aligned} \quad (2.20)$$

In the (Gaussian) free field theory, the composite fields (2.10) are spinless with scale dimension :

$$D \equiv \Delta + \bar{\Delta} = 2\alpha^2 + 4. \quad (2.21)$$

For generic value of the coupling  $\beta$  some divergences arise in the VEVs of the fields (2.10) due to the perturbation in (1.1) with imaginary coupling. They are generally cancelled if we add specific counterterms which contain spinless local fields with cut-off dependent coefficients. For  $0 < \beta^2 < 1$  the perturbation becomes relevant and a finite number of lower scale dimension counterterms are then sufficient. However, this procedure is regularization scheme dependent, i.e. one can always add finite counterterms. For generic values of  $\alpha$  this ambiguity in the definition of the renormalized expression for the fields (2.10) can be eliminated by fixing their scale dimensions to be (2.21). It exists however a set of values of  $\alpha$  for which the ambiguity still remains, but here we will not consider these isolated cases. In the BD model with imaginary coupling, this situation arises if two fields, say  $\mathcal{O}_\alpha$  and  $\mathcal{O}_{\alpha'}$ , satisfy the resonance condition :

$$D_\alpha = D_{\alpha'} + 2n(1-\beta^2) + 2m(1-\frac{\beta^2}{4}) \quad \text{with} \quad (n, m) \in \mathbb{N} \quad (2.22)$$

associated with the ambiguity :

$$\mathcal{O}_\alpha \longrightarrow \mathcal{O}_\alpha + \mu^n \mu'^m \mathcal{O}_{\alpha'} . \quad (2.23)$$

In this specific case one says that the renormalized field  $\mathcal{O}_\alpha$  has an  $(n|m)$ -th *resonance*<sup>3</sup> [12] with the field  $\mathcal{O}_{\alpha'}$ . Due to the condition (2.19) and using (2.21) we find immediatly that a resonance can appear between the descendent field  $(\partial\varphi)^2(\bar{\partial}\varphi)^2 e^{i\alpha\varphi}$  and the following primary fields :

$$\begin{aligned} (i) \quad & e^{i(\alpha+\beta)\varphi} \quad \text{i.e. } (n|m) = (1|0) \quad \text{for } \alpha = \frac{1}{2\beta} ; \\ (ii) \quad & e^{i(\alpha+2\beta)\varphi} \quad \text{i.e. } (n|m) = (2|0) \quad \text{for } \alpha = -\frac{\beta}{2} ; \\ (iii) \quad & e^{i(\alpha-\beta)\varphi} \quad \text{i.e. } (n|m) = (0|2) \quad \text{for } \alpha = \frac{\beta}{4} ; \\ (iv) \quad & e^{i(\alpha+\frac{\beta}{2})\varphi} \quad \text{i.e. } (n|m) = (1|1) \quad \text{for } \alpha = \beta . \end{aligned} \quad (2.24)$$

If we now look at the expression (2.15), we notice that the contribution (2.16), brought by the second level descendent field, and that of any of the exponential fields in (i), (ii), (iii) and (iv), have the same power behaviour in  $r$  ( $r^{4\alpha_1\alpha_2+4}$ ) at short-distance for the corresponding values of  $\alpha$  in (2.24). The integrals which appear in these contributions and their corresponding poles are, respectively :

$$\begin{aligned} j_1(\alpha_1\beta, \alpha_2\beta, \beta^2) & \quad \text{with the pole } \alpha = \frac{1}{2\beta} ; \\ j_2(\alpha_1\beta, \alpha_2\beta, \beta^2) & \quad \text{with the pole } \alpha = -\frac{\beta}{2} ; \\ j_2(-\frac{\alpha_1\beta}{2}, -\frac{\alpha_2\beta}{2}, \frac{\beta^2}{4}) & \quad \text{with the pole } \alpha = \frac{\beta}{4} ; \\ \mathcal{F}_{1,1}(\alpha_1\beta, \alpha_2\beta, \beta^2) & \quad \text{with the pole } \alpha = \beta . \end{aligned} \quad (2.25)$$

By analogy with the SG (or ShG) model, one expects that the VEV (2.16) (and similarly for the real coupling case) exhibits, at least, the same poles in order that the divergent contributions compensate each other. This last requirement leads for instance to the relations :

$$\begin{aligned} (i') \quad & \mathcal{R}es_{\alpha=\frac{1}{2\beta}} \mathcal{H}(\alpha) = 8\pi\beta^3 \mu \frac{\mathcal{G}_{\alpha+\beta}}{\mathcal{G}_\alpha} \Big|_{\alpha=\frac{1}{2\beta}} ; \\ (ii') \quad & \mathcal{R}es_{\alpha=-\frac{\beta}{2}} \mathcal{H}(\alpha) = -32\pi^2 \beta^3 \mu^2 \frac{\gamma(2\beta^2)}{\gamma(\beta^2)} \gamma(-1-\beta^2) \frac{\mathcal{G}_{\alpha+2\beta}}{\mathcal{G}_\alpha} \Big|_{\alpha=-\frac{\beta}{2}} ; \\ (iii') \quad & \mathcal{R}es_{\alpha=\frac{\beta}{4}} \mathcal{H}(\alpha) = 4\pi^2 \beta^3 \mu'^2 \frac{\gamma(\beta^2/2)}{\gamma(\beta^2/4)} \gamma(-1-\beta^2/4) \frac{\mathcal{G}_{\alpha-\beta}}{\mathcal{G}_\alpha} \Big|_{\alpha=\frac{\beta}{4}} ; \\ (iv') \quad & \mathcal{R}es_{\alpha=\beta} \mathcal{H}(\alpha) = -\frac{4}{(\alpha_1\alpha_2)^2} \mu \mu' \frac{\mathcal{G}_{\alpha+\frac{\beta}{2}}}{\mathcal{G}_\alpha} \Big|_{\alpha=\beta} \mathcal{R}es_{\alpha=\beta} \mathcal{F}_{1,1}(\alpha_1\beta, \alpha_2\beta, \beta^2) . \end{aligned} \quad (2.26)$$

These last conditions will be used in the next section to fix the normalization of the VEV (2.16). Let us now turn to the evaluation of (2.16) which plays an important role in the two-point function (2.15).

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<sup>3</sup>The same situation arise in any more general affine Toda theory.

### 3 Reflection relations and descendent fields

The BD model (1.1) can be regarded as two different perturbations of the Liouville field theory [5]. First, one can consider the Liouville action :

$$\mathcal{A}_L^{(1)} = \int d^2x \left[ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu e^{b\varphi} \right]. \quad (3.1)$$

The perturbation is then identified with  $e^{-\frac{b}{2}\varphi}$ . Alternatively, we can take :

$$\mathcal{A}_L^{(2)} = \int d^2x \left[ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu' e^{-\frac{b}{2}\varphi} \right] \quad (3.2)$$

as the initial action and consider  $e^{b\varphi}$  as a perturbation. Using the first picture, the holomorphic stress-energy tensor :

$$T(z) = -\frac{1}{4} (\partial\varphi)^2 + \frac{Q}{2} \partial^2 \varphi \quad (3.3)$$

ensures the local conformal invariance of the Liouville field theory (3.1) and similarly for the anti-holomorphic part. The exponential fields  $e^{a\varphi}$  are spinless primary fields with conformal dimension :

$$\Delta = a(Q - a). \quad (3.4)$$

The property of reflection relations which relates operators with the same quantum numbers is a characteristic of the CFT. Using the CPT framework, one expects that similar relations are also satisfied in the perturbed case (1.1). With the change  $b \rightarrow -b/2$  in (3.3) and using the second picture (3.2), one assumes that the VEV of the exponential field  $\langle e^{a\varphi} \rangle_{BD}$  satisfies simultaneously the following two functional equations [5] :

$$\begin{aligned} \langle e^{a\varphi} \rangle_{BD} &= R(a) \langle e^{(Q-a)\varphi} \rangle_{BD} ; \\ \langle e^{-a\varphi} \rangle_{BD} &= R'(a) \langle e^{(-Q'+a)\varphi} \rangle_{BD} \end{aligned} \quad (3.5)$$

with

$$Q = \frac{1}{b} + b \quad \text{and} \quad Q' = \frac{2}{b} + \frac{b}{2}. \quad (3.6)$$

The functions  $R(a)$ ,  $R'(a)$  are called “reflection amplitudes”. An exact expression for  $R(a)$  was presented in [6].  $R'(a)$  is obtained from  $R(a)$  by the substitutions  $b \rightarrow \frac{b}{2}$  and  $\mu \rightarrow \mu'$ . Under certain assumptions about the analytic properties of the VEV, the system (3.5) was solved and the VEV for these exponential fields was derived in [5].

Let us denote the descendent fields :

$$L_{[n]} \bar{L}_{[m]} e^{a\varphi} \equiv L_{-n_1} \dots L_{-n_1} \bar{L}_{-m_1} \dots \bar{L}_{-m_K} e^{a\varphi} \quad (3.7)$$

where  $[n] = [-n_1, \dots, -n_N]$  and  $[m] = [-m_1, \dots, -m_K]$  are arbitrary strings and  $L_n$ ,  $\bar{L}_n$  are the standard Virasoro generators :

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad \text{and} \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}. \quad (3.8)$$



The descendent fields (3.7) and the ones obtained after the reflection  $a \rightarrow Q - a$  possess the same quantum numbers. Consequently, using the arguments of [5, 12] based on the CPT framework, one also expects that their VEVs in the perturbed theory (1.1) satisfy the following “reflection relation” :

$$< L_{[n]} \bar{L}_{[m]} e^{a\varphi} >_{BD} = R(a) < L_{[n]} \bar{L}_{[m]} e^{(Q-a)\varphi} >_{BD} . \quad (3.9)$$

However, it is more convenient to use the basis :

$$(\partial^{n_1} \varphi) \dots (\partial^{n_N} \varphi) (\bar{\partial}^{m_1} \varphi) \dots (\bar{\partial}^{m_K} \varphi) e^{a\varphi} . \quad (3.10)$$

The main reason is that in (3.9) the components  $L_n, \bar{L}_n$  of the modified stress-tensor depend on  $a$ . Using (2.2) one can always express (3.7) in the basis (3.10). For our purpose we will need the relation [6] :

$$L_{-2} \bar{L}_{-2} e^{a\varphi} = \left[ -\frac{1}{4}(\partial\varphi)^2 + \left(\frac{Q}{2} + a\right)\partial^2\varphi \right] \left[ -\frac{1}{4}(\bar{\partial}\varphi)^2 + \left(\frac{Q}{2} + a\right)\bar{\partial}^2\varphi \right] e^{a\varphi} . \quad (3.11)$$

Furthermore, using (2.2) it implies :

$$< L_{-2} \bar{L}_{-2} e^{a\varphi} >_{BD} = \frac{1}{16} (1 + 2a(Q + 2a))^2 < (\partial\varphi)^2 (\bar{\partial}\varphi)^2 e^{a\varphi} >_{BD} \quad (3.12)$$

which leads to the following reflection relation :

$$\begin{aligned} (1 + 2a(Q + 2a))^2 < (\partial\varphi)^2 (\bar{\partial}\varphi)^2 e^{a\varphi} >_{BD} = \\ (1 + 2(Q - a)(3Q - 2a))^2 < (\partial\varphi)^2 (\bar{\partial}\varphi)^2 e^{(Q-a)\varphi} >_{BD} \end{aligned} \quad (3.13)$$

One can also consider the second picture (3.2) where the Liouville theory has coupling  $-\frac{b}{2}$  instead of  $b$  and is perturbed by  $e^{b\varphi}$ . If we define the analytic continuation of (2.16) :

$$H(a) = \frac{< (\partial\varphi)^2 (\bar{\partial}\varphi)^2 e^{a\varphi} >_{BD}}{< e^{a\varphi} >_{BD}} , \quad (3.14)$$

then the two different pictures provide us the following two functional relations :

$$\begin{aligned} H(a) &= \left[ \frac{(2b + 3/b - 2a)(3b + 2/b - 2a)}{(b + 2a)(1/b + 2a)} \right]^2 H(Q - a), \\ H(-a) &= \left[ \frac{(b + 6/b - 2a)(3b/2 + 4/b - 2a)}{(b/2 + 2a)(2/b + 2a)} \right]^2 H(-Q' + a). \end{aligned} \quad (3.15)$$

Notice that these equations are invariant with respect to the symmetry  $b \rightarrow -\frac{2}{b}$  with  $a \rightarrow -a$  in agreement with the well-known self-duality of the BD-model.

As was shown in the previous section, the solution of these functional equations should exhibits, at least, the poles (2.25) through the identification  $b = i\beta$  and  $a = i\alpha$ . Since the solution of (3.15) is defined up to a multiplication constant, we naturally choose to fix it

by imposing eqs. (2.26). We find that the “minimal” solution which follows from these constraints is :

$$\begin{aligned}
H(a) &= - \left[ \frac{m\Gamma(\frac{b^2}{h})\Gamma(\frac{2}{h})}{\Gamma(\frac{1}{3})\sqrt{3} 2^{2/3+3/2}(Q+Q')^2} \right]^4 \times \frac{\gamma^2(\frac{1}{3})}{\gamma(\frac{2b^2}{h})\gamma(\frac{4}{h})} \\
&\times \gamma(\frac{2ba+4}{h})\gamma(\frac{-2ba-b^2}{h})\gamma(\frac{2ba+3+b^2}{h})\gamma(\frac{-2ba-1}{h}) \\
&\times \gamma(\frac{-2ba+2b^2}{h})\gamma(\frac{2ba-2}{h})\gamma(\frac{-2ba+2+3b^2/2}{h})\gamma(\frac{2ba-b^2/2}{h})
\end{aligned} \tag{3.16}$$

where  $h = 6 + 3b^2$  is the “deformed” Coxeter number [19, 20]. Here we have used the exact relation between the parameters  $\mu$  and  $\mu'$  in the action (1.1) and the mass of the particle  $m$  [5] :

$$m = \frac{2\sqrt{3}\Gamma(1/3)}{\Gamma(1+b^2/h)\Gamma(2/h)} (-\mu\pi\gamma(1+b^2))^{1/h} (-2\mu'\pi\gamma(1+b^2/4))^{2/h}. \tag{3.17}$$

It is easy to see that for  $b = i\beta$  and  $a = i\alpha$ ,  $\mathcal{H}(\alpha)$  possess poles located at :

$$\alpha_0 + q\left(\frac{3}{\beta} - \frac{3\beta}{2}\right), \quad q \in \mathbb{Z} \quad \text{with} \quad \alpha_0 \in \left\{-\frac{\beta}{2}, \frac{1}{2\beta}, \frac{\beta}{4}, \beta, \frac{3}{2\beta} - \frac{\beta}{2}, -\frac{1}{\beta}, -\frac{1}{\beta} + \frac{3\beta}{4}, \frac{2}{\beta}\right\}.$$

But as long as we consider  $\alpha$  that satisfy (2.19) there remain<sup>4</sup> the expected poles (2.25) :

$$\alpha_0 \in \left\{-\frac{\beta}{2}, \frac{1}{2\beta}, \frac{\beta}{4}, \beta\right\}. \tag{3.18}$$

It is well-known that the Bullough-Dodd model at the specific value of the coupling  $b^2 = 2$  and the sinh-Gordon model at  $b^2 = \frac{1}{2}$  give an equivalent Lagrangian representation of the same QFT. Then, as expected, one can check that (3.16) evaluated at  $b^2 = 2$  coincides exactly with the same quantity in the ShG model evaluated at  $b^2 = \frac{1}{2}$ .

Accepting the conjecture (3.16) and using eq. (3.12) for  $a = 0$  one can easily deduce for instance :

$$\langle T\bar{T} \rangle_{BD} \equiv \langle L_{-2}\bar{L}_{-2}\mathbb{I} \rangle_{BD} = -\pi^2 f_{BD}^2 \tag{3.19}$$

where

$$f_{BD} = \frac{m^2}{16\sqrt{3}\sin(\frac{\pi b^2}{h})\sin(\frac{2\pi}{h})} \tag{3.20}$$

is the bulk free energy of the BD model [5].

---

<sup>4</sup>Notice that for  $\frac{2}{3} < \beta^2 < 1$  one also has the pole  $\alpha'_0 \equiv -\frac{1}{\beta} + \frac{3\beta}{4}$ . However, this one is nothing but obtained from the reflection  $\alpha'_0 = (\beta - \frac{1}{\beta}) - \frac{\beta}{4}$ .

## 4 Comparison with the semi-classical results

As we saw previously, the OPE proposed in eq. (2.15) plays a crucial role in the determination of the prefactor of the  $\alpha$ -dependent part of  $\mathcal{H}(\alpha)$ , using eqs. (2.26). It is therefore important to check this expression, using for instance the semi-classical expansion. In what follows, we will compare (2.15) with the semi-classical calculations based on the action (1.1).

Let us consider (2.15) for  $\alpha_1 = \sigma/\beta$ ,  $\alpha_2 = \omega\beta$  in the classical limit  $\beta \rightarrow 0$ . Then the saddle-point evaluation of the functional integral based on the action (1.1) leads to the field configuration  $\varphi_{cl} = \frac{2}{i\beta}(\phi(t) + \frac{1}{3}\log(\frac{\mu'}{2\mu}))$ ,  $t = \frac{mr}{2\sqrt{3}}$ , where  $\phi(t)$  is a solution of the classical Bullough-Dodd equation :

$$\partial_t^2 \phi + t^{-1} \partial_t \phi = 4(e^{2\phi} - e^{-\phi}) \quad (4.1)$$

with the following asymptotic conditions :

$$\begin{aligned} \phi(t) &\rightarrow -A \log t - \log B + o(1) & \text{as } t \rightarrow 0, \\ \text{and } \phi(t) &\rightarrow -\frac{4\sqrt{3}}{\pi} \sin\left(\frac{\pi\sigma}{3}\right) \sin\left(\frac{\pi(1+\sigma)}{3}\right) K_0(2\sqrt{3}t) & \text{as } t \rightarrow \infty. \end{aligned} \quad (4.2)$$

Here we denoted :

$$A \equiv -2\sigma; \quad B \equiv \frac{3^{2\sigma}}{\gamma(\frac{1-\sigma}{3})\gamma(\frac{2-2\sigma}{3})}, \quad (4.3)$$

and  $K_0(x)$  is the MacDonald function. Such a solution was considered in [21].

Taking into account the above considerations, the two-point function takes the following form in the semi-classical limit :

$$\frac{\langle e^{i\omega\beta\varphi}(x) e^{i\frac{\sigma}{\beta}\varphi}(y) \rangle_{BD}}{\langle e^{i\omega\beta\varphi} \rangle_{BD} \langle e^{i\frac{\sigma}{\beta}\varphi} \rangle_{BD}} \Big|_{\beta^2 \rightarrow 0} = (e^{-2\phi})^{-\omega}. \quad (4.4)$$

Following [21] it is not difficult to obtain the first few terms in the  $t \rightarrow 0$  expansion :

$$\begin{aligned} e^{-2\phi(t)} = & B^2 t^{2A} \left( 1 - \frac{2}{B^2(A-1)^2} t^{2-2A} + \frac{8B}{(A+2)^2} t^{2+A} + \frac{40B^2}{(A+2)^4} t^{4+2A} + \right. \\ & \left. + \frac{8(5A^2 - 4A - 28)}{(A-1)^2(A+2)^2(A-4)^2 B} t^{4-A} + \frac{1}{(A-1)^4 B^4} t^{4-4A} + O(t^{6+3A}, t^6) \right). \end{aligned} \quad (4.5)$$

We would like now to compare these results with the corresponding limit of (2.15). First, using the result for the exact VEV in the Bullough-Dodd model (2.18) proposed in [5] we obtain the following ratios :

$$\begin{aligned} \frac{\mathcal{G}_{\sigma/\beta+\omega\beta}}{\mathcal{G}_{\sigma/\beta}\mathcal{G}_{\omega\beta}} &= m^{4\omega\sigma} \frac{\left(\gamma(\frac{1-\sigma}{3})\gamma(\frac{2-2\sigma}{3})\right)^{2\omega}}{2^{4\omega\sigma} 3^{6\omega\sigma}}; \\ \mu^n \frac{\mathcal{G}_{\sigma/\beta+\omega\beta+n\beta}}{\mathcal{G}_{\sigma/\beta}\mathcal{G}_{\omega\beta}} &= \frac{m^{4\omega\sigma+4n\sigma+2n}}{\beta^{2n} \pi^n} \frac{\left(\gamma(\frac{1-\sigma}{3})\gamma(\frac{2-2\sigma}{3})\right)^{2\omega+2n}}{2^{4\omega\sigma+4n\sigma+2n} 3^{6\omega\sigma+6n\sigma+n}}; \end{aligned} \quad (4.6)$$

$$\mu^n \frac{\mathcal{G}_{\sigma/\beta+\omega\beta-n\beta/2}}{\mathcal{G}_{\sigma/\beta}\mathcal{G}_{\omega\beta}} = \frac{m^{4\omega\sigma-2n\sigma+2n}}{\beta^{2n}\pi^n} \frac{\left(\gamma\left(\frac{1-\sigma}{3}\right)\gamma\left(\frac{2-2\sigma}{3}\right)\right)^{2\omega-n}}{2^{4\omega\sigma-2n\sigma+n}3^{6\omega\sigma-3n\sigma+n}};$$

$$\mu^n \mu' \frac{\mathcal{G}_{\sigma/\beta+\omega\beta+(n-1/2)\beta}}{\mathcal{G}_{\sigma/\beta}\mathcal{G}_{\omega\beta}} = \frac{m^{4\omega\sigma+4n\sigma-2\sigma+2n+2}}{\beta^{2n+2}\pi^{n+1}} \frac{\left(\gamma\left(\frac{1-\sigma}{3}\right)\gamma\left(\frac{2-2\sigma}{3}\right)\right)^{2\omega+2n-1}}{2^{4\omega\sigma+(4n-2)\sigma+2n+1}3^{6\omega\sigma+(6n-3)\sigma+n+1}}.$$

Furthermore, the mass- $\mu$  relation (3.17) proposed in [5] gives :

$$\mu\mu'^2 \xrightarrow{\beta^2 \rightarrow 0} \left(\frac{m}{2\sqrt{3}}\right)^6 \frac{4}{\pi^3\beta^6} \quad (4.7)$$

whereas, using eq. (2.9) and  $\mathcal{F}_{1,1}(\alpha_1\beta, \alpha_2\beta, \beta^2)$  which can be deduced from the results of [22], we have :

$$j_n(\sigma, \omega\beta^2, \beta^2) \xrightarrow{\beta^2 \rightarrow 0} \frac{\pi^n \beta^{2n}}{n!} \frac{\Gamma(2\omega+n)}{(1+2\sigma)^{2n}} \frac{\Gamma(2\omega)}{\Gamma(2\omega)};$$

$$j_n(-\sigma/2, -\omega\beta^2/2, \beta^2/4) \xrightarrow{\beta^2 \rightarrow 0} \frac{\pi^n \beta^{2n}}{n!} \frac{\Gamma(-4\omega+n)}{2^{2n}(1-\sigma)^{2n}} \frac{\Gamma(-4\omega)}{\Gamma(-4\omega)};$$

$$\mathcal{F}_{1,1}(\sigma, \omega\beta^2, \beta^2) \xrightarrow{\beta^2 \rightarrow 0} -\frac{\pi^2 \beta^4 \omega}{(1+2\sigma)^2(1-\sigma)^2(2+\sigma)^2} [\sigma^2(2\omega+7) + \sigma(8\omega+10) + 8\omega+1];$$

$$\mathcal{H}(\sigma/\beta + \omega\beta) \xrightarrow{\beta^2 \rightarrow 0} O(\beta^4).$$
(4.8)

If we now use the same notations as above (4.3), the semi-classical limit of the expression (2.15) takes the following form :

$$\frac{\mathcal{G}_{\sigma/\beta, \omega\beta}}{\mathcal{G}_{\sigma/\beta}\mathcal{G}_{\omega\beta}} \xrightarrow{\beta^2 \rightarrow 0} t^{-2A\omega} B^{-2\omega} \times \left[ 1 + \frac{2\omega}{B^2(1-A)^2} t^{2-2A} - \frac{8\omega B}{(2+A)^2} t^{2+A} + \frac{\omega(2\omega+1)}{B^4(1-A)^4} t^{4-4A} \right. \\ \left. + \frac{8\omega(4\omega-1)B^2}{(2+A)^4} t^{4+2A} - \frac{8\omega(A^2(2\omega+7) - A(16\omega+20) + 32\omega+4)}{B(1-A)^2(2+A)^2(4-A)^2} t^{4-A} \right. \\ \left. + O(t^6) \right]. \quad (4.9)$$

It is straightforward to check that this result agrees perfectly with (4.4) through (4.5).

## 5 Expectation values of the descendent fields in $\Phi_{12}$ , $\Phi_{21}$ and $\Phi_{15}$ perturbed minimal models

For imaginary value of the coupling  $b = i\beta$ ,  $\mu \rightarrow -\mu$  and  $\mu' \rightarrow -\mu'$  the action of the BD model (1.1) becomes complex. Whereas it is not clear if it can be defined as a QFT, this model is known to be integrable and its  $S$ -matrix was constructed in [14]. It is known that this model possess a quantum group symmetry  $U_q(A_2^{(2)})$  with deformation parameter  $q = e^{\frac{i\pi}{\beta^2}}$  [14, 15]. An important role is played by one of its subalgebras  $U_q(sl_2) \subset U_q(A_2^{(2)})$ . Following [14] (see also [5]), we can restrict the Hilbert space of states of the complex BD

model at special values of the coupling constant, more precisely when  $q$  is a root of unity, i.e. for :

$$\beta^2 = \frac{p}{p'} \quad \text{or} \quad \beta^2 = \frac{p'}{p} \quad \text{with} \quad 1 < p < p' \quad (5.1)$$

relative prime integers, in which case the complex BD is identified with the perturbed minimal models (1.2) or (1.3), respectively. In the following,  $\Phi_{lk}$  will denote a primary field of the minimal model  $\mathcal{M}_{p/p'}$ .

In the first case, the exact relation between the parameters  $\lambda$  in (1.2) and the mass of the fundamental kink  $M$  can be found in [5] with the result :

$$\lambda^2 = \frac{\pi^{\frac{\xi+4}{\xi+1}} \Gamma^2(\frac{\xi}{4\xi+4}) \Gamma(\frac{1}{2} - \frac{1}{\xi+1})}{2^{\frac{\xi+5}{\xi+1}} \Gamma^2(\frac{3\xi+4}{4\xi+4}) \Gamma(\frac{1}{2} + \frac{1}{\xi+1})} \left[ \frac{M \Gamma(\frac{2\xi+2}{3\xi+6})}{\sqrt{3} \Gamma(\frac{1}{3}) \Gamma(\frac{\xi}{3\xi+6})} \right]^{\frac{3\xi+6}{\xi+1}}. \quad (5.2)$$

Here we denote

$$\xi = \frac{p}{p' - p}. \quad (5.3)$$

For unitary minimal models  $\xi > 1$  which, for  $\Im m(\lambda) = 0$ , corresponds to the massive phase [5]. Using the particle-breather identification :

$$m = 2M \sin\left(\frac{\pi\xi}{3\xi+6}\right) \quad (5.4)$$

and eqs. (3.12), (3.16) for imaginary coupling  $b = i\beta$  and parameter  $a = i(\frac{l-1}{2\beta} - \frac{k-1}{2}\beta)$ , it is straightforward to get the VEV in the model associated with the action (1.2) :

$$\frac{\langle 0_s | L_{-2} \bar{L}_{-2} \Phi_{lk} | 0_s \rangle}{\langle 0_s | \Phi_{lk} | 0_s \rangle} = - \left[ \frac{\sqrt{3} \pi (\xi + 2) M \Gamma(1 + \frac{2+2\xi}{3\xi+6})}{\Gamma(\frac{1}{3}) 2^{2/3+1/2} \Gamma(\frac{\xi}{3\xi+6})} \right]^4 \frac{\gamma^2(1/3)}{\gamma(-\frac{2\xi}{3\xi+6}) \gamma(\frac{4+4\xi}{3\xi+6})} \times \mathcal{W}_{12}((\xi+1)l - \xi k) \quad (5.5)$$

with

$$\mathcal{W}_{12}(\eta) = \frac{1}{\xi^2 (\xi+1)^2} \times w(\eta; 5+4\xi, 4+2\xi, -1-2\xi, 1+\xi/2; 3\xi+6)$$

where we introduce the useful notation :

$$w(\eta; a_1, a_2, a_3, a_4; g) = \prod_{i=1}^4 \gamma\left(\frac{a_i + \eta}{g}\right) \gamma\left(\frac{a_i - \eta}{g}\right).$$

Here  $|0_s\rangle$  is one of the degenerate ground states of the QFT (1.2) (see [5] for a detailed discussion of the vacuum structure of the model). Taking  $\Phi_{lk}$  in (5.5) to be the identity operator, it is easy to get :

$$\langle T \bar{T} \rangle = - \frac{\pi^2 M^4}{48} \frac{\sin^2(\frac{\pi\xi}{3\xi+6})}{\sin^2(\frac{\pi(2\xi+2)}{3\xi+6})}. \quad (5.6)$$

A simple check consists to consider the scaling Lee-Yang model which corresponds to  $p = 2$ ,  $p' = 5$  i.e.  $\xi = \frac{2}{3}$  in (1.2). As  $\Phi_{12} \equiv \Phi_{13}$  for these values, we must obtain the result of [12]. Using (5.4) the lightest mass in (1.2) is :

$$m = 2M \sin\left(\frac{\pi}{12}\right)$$

Replacing this expression in (5.5) for  $l = 1$ ,  $k = 3$  and (5.6) it is easy to see that the results are in perfect agreement with the ones of [12].

In the second restriction  $\beta^2 = p'/p$ , which leads to the action (1.3). The exact relation between the parameter  $\hat{\lambda}$  and the mass of the fundamental kink  $M$  is in this case [5] :

$$\hat{\lambda}^2 = \frac{\pi^{\frac{\xi-3}{\xi}}}{2^{\frac{4-\xi}{\xi}}} \frac{\Gamma^2(\frac{\xi+1}{4\xi})\Gamma(\frac{1}{2} + \frac{1}{\xi})}{\Gamma^2(\frac{3\xi-1}{4\xi})\Gamma(\frac{1}{2} - \frac{1}{\xi})} \left[ \frac{M\Gamma(\frac{2\xi}{3\xi-3})}{\sqrt{3}\Gamma(\frac{1}{3})\Gamma(\frac{\xi+1}{3\xi-3})} \right]^{\frac{3\xi-3}{\xi}}. \quad (5.7)$$

Along the same line as for the  $\Phi_{12}$  perturbation we obtain the following expression for the VEV in the model associated with the action (1.3) :

$$\frac{\langle 0_s | L_{-2} \bar{L}_{-2} \Phi_{lk} | 0_s \rangle}{\langle 0_s | \Phi_{lk} | 0_s \rangle} = - \left[ \frac{\sqrt{3}\pi(1-\xi)M\Gamma(1 - \frac{2\xi}{3-3\xi})}{\Gamma(\frac{1}{3})2^{2/3+1/2}\Gamma(-\frac{\xi+1}{3-3\xi})} \right]^4 \frac{\gamma^2(1/3)}{\gamma(\frac{2\xi+2}{3-3\xi})\gamma(\frac{-4\xi}{3-3\xi})} \times \mathcal{W}_{21}((\xi+1)l - \xi k) \quad (5.8)$$

with

$$\mathcal{W}_{21}(\eta) = \frac{1}{\xi^2(\xi+1)^2} \times w(\eta; 1-4\xi, 2-2\xi, 1+2\xi, 1/2 - \xi/2; 3-3\xi)$$

where  $|0_s\rangle$  is one of the degenerate ground states [5] of the QFT (1.3). The analog of the formula (5.6) is now :

$$\langle T\bar{T} \rangle = - \frac{\pi^2 M^4}{48} \frac{\sin^2(\frac{\pi(1+\xi)}{3-3\xi})}{\sin^2(\frac{2\pi\xi}{3-3\xi})}. \quad (5.9)$$

Another subalgebra of  $U_q(A_2^{(2)})$  is the subalgebra  $U_q(sl_2)$ . One can again restrict the phase space of the complex BD with respect to this subalgebra for a special value of the coupling :

$$\beta^2 = \frac{4p}{p'} \quad \text{with} \quad 2p < p' \quad (5.10)$$

relative prime integers. Then, for this value of the coupling, the BD model is identified with the perturbed minimal model with the action (1.4). The exact relation between  $\hat{\lambda}$  and the mass<sup>5</sup>  $m$  is [5] :

$$\tilde{\lambda}^2 = \frac{(1+\xi)^2}{2\pi(1-4\xi)(1-2\xi)} \frac{\Gamma(\frac{4\xi}{1+\xi})}{\Gamma(\frac{3-\xi}{1+\xi})} \sqrt{\frac{\Gamma(\frac{1}{1+\xi})\Gamma(\frac{5\xi}{1+\xi})}{\Gamma(\frac{\xi}{1+\xi})\Gamma(\frac{1-4\xi}{1+\xi})}} \left[ \frac{m\Gamma(\frac{3-5\xi}{3-3\xi})\Gamma(\frac{1+\xi}{3-3\xi})}{2\sqrt{3}\Gamma(\frac{1}{3})} \right]^{\frac{6(1-\xi)}{1+\xi}}. \quad (5.11)$$

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<sup>5</sup>The general vacuum structure in the model (1.4) is not clearly understood. However it is expected that it possesses particles and kinks similarly to the other models (see for instance refs. [17] for details). The physical mass scale  $m$  is then associated with one of its particles [5].

Here we keep the definition (5.3). In particular, the massive phase corresponds to :

$$\begin{aligned} 0 < \xi < \frac{1}{4}, & \quad \Im m \tilde{\lambda} = 0 ; \\ \frac{1}{4} < \xi < \frac{3}{5}, & \quad \Re e \tilde{\lambda} = 0 . \end{aligned}$$

As for the two previous cases one would like to obtain the expectation values of the descendent fields for *any* primary operator  $\Phi_{lk}$ . For  $l > 1$  these fields are not invariant with respect to the subalgebra  $U_{q^4}(sl_2)$  on the contrary to  $\Phi_{1k}$ . However, one expects that they only differ by a c-number coefficient characterizing the degenerate structure of the vacua  $|0_s\rangle$ . Taking the ratio of the VEV of the descendent field of  $\Phi_{lk}$  associated with the action (1.4) and the VEV of the primary field itself, one obtains :

$$\begin{aligned} \frac{\langle 0_s | L_{-2} \bar{L}_{-2} \Phi_{lk} | 0_s \rangle}{\langle 0_s | \Phi_{lk} | 0_s \rangle} &= - \left[ \frac{m \xi \Gamma(1 + \frac{1+\xi}{3-3\xi}) \Gamma(-\frac{2\xi}{3-3\xi})}{\Gamma(\frac{1}{3}) \sqrt{3} 2^{2/3+1/2}} \right]^4 \frac{\gamma^2(1/3)}{\gamma(-\frac{4\xi}{3-3\xi}) \gamma(\frac{2+2\xi}{3-3\xi})} \\ &\quad \times \mathcal{W}_{15}((\xi+1)l - \xi k) \end{aligned} \quad (5.12)$$

with

$$\mathcal{W}_{15}(\eta) = \frac{1}{\xi^2(\xi+1)^2} \times w(\eta; \xi+5, 4-4\xi, -1-5\xi, 1-\xi; 6-6\xi) .$$

In particular, we have :

$$\langle T \bar{T} \rangle = - \frac{\pi^2}{768} \frac{m^4}{\sin^2(\frac{2\pi\xi}{3-3\xi}) \sin^2(\frac{\pi(1+\xi)}{3-3\xi})} . \quad (5.13)$$

## 6 Concluding remarks

In conclusion, we proposed in this paper an exact expression for the VEV of the second level descendent of the exponential field  $\langle (\partial\varphi)^2 (\bar{\partial}\varphi)^2 e^{i\alpha\varphi} \rangle$  in the BD model. The calculation is based on the so-called “reflection relations” which lead to a system of functional equations for this VEV. While the solution is not unique, we chose the “minimal solution” obeying some residue conditions. By performing a quantum group restriction in the case of complex BD model we found also the VEVs of the descendants of the primary fields in the perturbed minimal CFT models (1.2), (1.3), (1.4).

It is rather interesting to notice that in eq. (3.19), the exact VEV  $\langle T \bar{T} \rangle_{BD}$  is simply related to the VEV of the trace of the energy momentum tensor :

$$\Theta = \frac{1}{4} T_\nu^\nu = \pi(1 - \Delta_{pert}) \mu \Phi_{pert}$$

where  $1 - \Delta_{pert} = h/4$  and  $\Phi_{pert} = e^{-\frac{h}{2}\varphi}$  as follows :

$$\langle T \bar{T} \rangle_{BD} = - \langle \Theta \rangle_{BD}^2 . \quad (6.1)$$

This was already noticed in [12] for the ShG case. We expect this property to be general, i.e. to be confirmed for other integrable theories. However, we have no proof yet of this phenomena.

We would like to notice two important differences between ShG and BD models. First, in the  $\beta^2$  expansion of the two-point function the quantity  $\mathcal{H}(\alpha)$  (2.16) comes with a coefficient of order  $\beta^2$ . Therefore, it cannot be checked directly in the semi-classical approximation, although the later is in agreement with the short distance expansion of the two-point function, thus giving a strong support to our conjecture (3.16). For a direct check one has to go beyond the classical limit and consider the first order in the perturbation theory based on the action (1.1).

Another difference is the appearing of the third level descendents in the OPE of the exponential fields. As a consequence, the following quantity appear in the short distance expansion of the two-point function :

$$\mathcal{K}(\alpha) = \frac{\langle (\partial\varphi)^3 (\bar{\partial}\varphi)^3 e^{i\alpha\varphi} \rangle_{BD}}{\langle e^{i\alpha\varphi} \rangle_{BD}}. \quad (6.2)$$

In contrast with  $\mathcal{H}(\alpha)$  it is sensitive to the semi-classical expansion - it combines with the integral  $\mathcal{F}_{1,2}$  in (2.15) in order to match the corresponding term coming from the semiclassical calculation based on the action (1.1). The function (6.2) can also be obtained using the “reflection relations” approach.

Using our results, one can deduce easily the next-to leading contributions in the short distance behaviour of the two-point functions of primary operators for different perturbed minimal models : for instance the Ising model in a magnetic field [23], the tricritical Ising model perturbed by its leading energy density operator (with conformal dimension  $\Delta_{12} = 1/10$ ) [22] or perturbed by its subleading magnetic operator (with conformal dimension  $\Delta_{21} = 7/16$ ), and so on.

Several models can be worked out along the same line, using the known results for the three-point functions of the CFT. For instance, the super ShG model or, more generally, the parafermionic ShG model [24, 8].

We intend to discuss these various questions in a forthcoming publication [25].

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